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The full group C*-algebra of the modular group is primitive

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Abstract

We show that the full group C*-algebra of $PSL(n, \mathbb{Z})$ is primitive when $n = 2$, and not primitive when $n \geq 3$.

Dedicated to the memory of Gerard J. Murphy.

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1 Introduction

In this note, all the groups we consider are supposed to be countable and discrete. Such a group G is called C*-simple if its reduced group C*-algebra $C_r^*(G)$ is simple. As the full group C*-algebra $C^*(G)$ is simple only when G is trivial, this terminology is not ambiguous. There has been a lot of interest in the class of C*-simple groups. For a recent exposition, with many references, the reader may consult [8], where P. de la Harpe explains how C*-simplicity may be regarded as an extreme case of non-amenability.

A weaker condition than C*-simplicity of G is primitivity of $C_r^*(G)$. We recall that a C*-algebra is called primitive if it has a *faithful* irreducible representation, and that primitivity is equivalent to primeness for separable C*-algebras ([15]). It is well known (see [13, 12]) that $C_r^*(G)$ is primitive if and only if G is ICC (that is, every non-trivial conjugacy class in G is infinite).

The problem of determining when C$^*(G)$ is primitive seems hard in general. It may be rephrased as follows: when is the universal unitary representation of G weakly equivalent to an irreducible unitary representation ?
A necessary condition is that G is ICC ([12]), and this condition is also sufficient when G is assumed to be amenable, as $C^*(G)$ is then *-isomorphic to $C_r^*(G)$. One should be aware that this problem is quite different from the one of determining the class of groups having a faithful irreducible unitary representation. This class contains many other groups besides all ICC groups (see [4]).

Until 2003, the only known non-amenable groups having a primitive full group C*-algebra were non-abelian free groups, as originally etablished by H. Yoshizawa [17] and rediscovered later by M.D. Choi [6]. In his investigation of this problem in [12], G.J. Murphy was able to exhibit many new examples of (non-amenable ICC) groups G for which C$^*(G)$ is primitive (see [12, Theorems 3.3 and 3.4]): G can be any group having a free product decomposition $G = F * Z$, where either

i) F is a non-trivial free group and Z is a non-trivial amenable group, or

ii) F is a non-abelian free group and Z is group such that $C^*(Z)$ admits no non-trivial projections.

In ii) one may for example take $Z = Z_1 * Z_2$ where both Z_1 and Z_2 are torsion-free amenable groups (see [12, Corollary 3.5]).

In [8, Problem 25], de la Harpe raises the problem of finding other (non-amenable ICC) groups having a primitive full group C*-algebra. We show in this paper (Theorem 1) that the modular group $\mathrm{PSL}(2,\mathbb{Z})$ is such a group.

Our proof uses the well known fact that $G = \mathrm{PSL}(2,\mathbb{Z})$ may be written as $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \,|\, a^2 = 1, b^3 = 1\rangle$ with $a = [A]$ and $b = [B]$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

If one realizes $PSL(2,\mathbb{Z})$ as a group of fractional linear transformations on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, then

$$a(z) = -\frac{1}{z}, \; b(z) = -\frac{1}{z} + 1\,.$$

An outline of our proof is as follows: Let H be the kernel of the canonical homomorphism from $G = \mathbb{Z}_2 * \mathbb{Z}_3$ onto $\mathbb{Z}_2 \times \mathbb{Z}_3$. Then H is a normal subgroup of G, which is known to be freely generated by $abab^2$ and ab^2ab (see e.g. [16]). Exploiting a certain phase-action of the circle group $\mathbb{T}$ on $C^*(H)$, we show that one can pick an irreducible faithful representation of $C^*(H)$ such that the induced representation of $C^*(G)$ is also faithful and irreducible. (In fact, we show that one can produce in this way a countably infinite family of unitarily inequivalent representations of $C^*(G)$). A similar idea was used by Murphy in his proof of [12, Theorem 3.3], where he considers certain semidirect products of non-abelian free groups by amenable groups. However, in our case, the exact sequence $1 \to H \to G \to \mathbb{Z}_2 \times \mathbb{Z}_3 \to 1$ does not split, so we have to decompose $C^*(G)$ as a twisted crossed product of $C^*(H)$ by $\mathbb{Z}_2 \times \mathbb{Z}_3$ and use results of J. Packer and I. Raeburn from [14]. Actually, when H is a normal subgroup of a group G, we give a criterion ensuring that primitivity of C$^*(H)$ passes over to C$^*(G)$ (see Theorem 2), and uses it to deduce Theorem 1.

Murphy mentions in [12] that he knows no example of an ICC group whose full group C*-algebra is not primitive, but suspects that $\mathbb{F}_2 \times \mathbb{F}_2$ is such an example. More generally, if $\mathbb{F}$ is a free non-abelian group, one may wonder whether $C^*(\mathbb{F} \times \mathbb{F})$ is primitive or not. Note that if it happens that $C^*(\mathbb{F} \times \mathbb{F})$ is *not* primitive, it will follow that

$$C^*(\mathbb{F}) \otimes_{max} C^*(\mathbb{F}) \not\simeq C^*(\mathbb{F}) \otimes_{min} C^*(\mathbb{F}).$$

Thus, when $\mathbb{F}$ has infinitely many generators, this would solve negatively an open problem of E. Kirchberg, which he has shown is equivalent to Connes' famous embedding problem (see [9]). It is therefore to be expected that proving non-primitivity of $C^*(\mathbb{F} \times \mathbb{F})$ won't be an easy task, if successfull at all.

In this note, we show that there does exist ICC groups whose full group C*-algebras are not primitive. We first observe (in Proposition 3) that if G has Kazhdan's property (T) (see e.g. [5]) and $C^*(G)$ is primitive, then G must be trivial. Hence, if we let G be any non-trivial ICC group having property (T), then $C^*(G)$ is not primitive. We may here for example take $G = PSL(n, \mathbb{Z})$ for any integer $n \geq 3$ (see [5]). Moreover, as it is known that $PSL(n, \mathbb{Z})$ is always C*-simple (see [1, 2, 8]), this also shows that C*-simplicity of G does not imply that $C^*(G)$ is primitive.

In view of the knowledge accumulated so far, a natural question is the following:

Assume that a group G may be written as a free product $G_1 * G_2$ for some non-trivial groups G_1 and G_2 not both of order 2. Is $C^*(G)$ primitive?

Very recently, we have etablished that the answer to this question is positive when both G_1 and G_2 are also assumed to be amenable. (See our Remark in the next section; details will appear in a subsequent paper). This supports our guess that the answer should always be positive.

2 Primitivity of full group C*-algebras and the modular group

We use standard notation and terminology in operator algebras, as found in [7], [15] and in any other standard textbook. All Hilbert spaces are assumed to be complex. By a representation of a C*-algebra, we always mean a $*$-homomorphism into the bounded operators $\mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. We use the same symbol $\simeq$ to denote unitary equivalence of operators on Hilbert spaces, unitary equivalence of representations of a C*-algebra and $*$-isomorphism between C*-algebras.

Our main purpose is show the following:

Theorem 1. *$C^*(PSL(2,\mathbb{Z}))$ is primitive. Moreover, there exists a countably infinite family of unitarily inequivalent irreducible faithful representations of $C^*(PSL(2,\mathbb{Z}))$.*

One possible approach to prove this result might be to look at the complementary series of $PSL(2,\mathbb{R})$ (see e.g. [10]) and restrict to $PSL(2,\mathbb{Z})$. But it is not clear to us that some of the irreducible representations one obtains in this way are faithful at the C*-level. Note that looking at the principal series or at the discrete series will definitely not work as the representations one then gets by restriction to the the modular group are known to be weakly equivalent to the regular representation (see [3]).

Our approach will be based on a certain permanence property for primitivity of full group C*-algebras. To state it in a conceptual manner, we introduce some notation and terminology.

Let A be a C*-algebra and $\widehat{A}$ denote the set of unitary equivalence classes of non-zero irreducible representations of A. Set

$$\widehat{A}^o = \{\, [\pi] \in \widehat{A} \mid \pi \text{ is faithful} \,\}\,.$$

This set is clearly well-defined, and non-empty if and only if A is primitive.

Assume now that a group G has a normal subgroup H such that $C^*(H)$ is primitive and set $K = G/H$. Then K acts on $\widehat{C^*(H)}^{\,o}$ in a natural way.

To see this, let $n : K \to G$ be a normalized section for the canonical homomorphism p from G onto K (so $n(e_K) = e_G$ and $p \circ n = id_K$).

Let $\alpha : K \to \mathrm{Aut}(C^*(H))$ and $u : K \times K \to C^*(H)$ be determined by

$$\alpha_k(i_H(h)) = i_H(\, n(k)\, h\, n(k)^{-1}\,), \quad k \in K,\, h \in H,$$

$$u(k,l) = i_H(\, n(k)\, n(l)\, n(kl)^{-1}), \quad k,l \in K,$$

where i_H denotes the canonical injection of H into $C^*(H)$.

Then (α, u) is a twisted action of K on $C^*(H)$ in the sense of J. Packer and I. Raeburn (see [14]); especially, we have

$$\alpha_k\, \alpha_l = \mathrm{Ad}(u(k,l))\, \alpha_{kl},\ \ k,l \in K,$$

where, as usual, $\mathrm{Ad}(v)$ denotes the inner automorphism implemented by some unitary v in $C^*(H)$.

This twisted action (α, u) clearly induces an action of K on $\widehat{C^*(H)}$ given by

$$k \cdot [\pi] = [\pi \circ \alpha_{k^{-1}}].$$

By restriction, we get an action of K on $\widehat{C^*(H)}^{\,o}$, which is easily seen to be independent of the choice of normalized section n for p.
We will call this action for *the natural action of* $K = G/H$ *on* $\widehat{C^*(H)}^{\,o}$.

We will also use the following definition:

Let a group K with identity e acts on a nonempty set X. Then we say that the action has a *free point* $x \in X$ whenever $k \cdot x \neq x$ for all $k \in K, k \neq e$.

Then the following result holds :

Theorem 2. *Assume that a group G has a normal subgroup H such that*

- *$C^*(H)$ is primitive,*
- *$K = G/H$ is amenable,*
- *the natural action of K on $\widehat{C^*(H)}^{\,o}$ has a free point.*

Then $C^*(G)$ is primitive.

Proof. We use the notation introduced above and note that Packer and Raeburn have shown (see [14, Theorem 4.1]) that $C^*(G)$ may be decomposed as the twisted crossed product associated with (α, u):

$$C^*(G) \simeq C^*(H) \times_{\alpha, u} K\,.$$

Let $[\pi] \in \widehat{C^*(H)}^{\,o}$ be a free point for the natural action of K. This means that we have

$$\pi \circ \alpha_k \not\simeq \pi \text{ for all } k \in K, k \neq e\,.$$

Now, this condition implies that the induced regular representation $\mathrm{Ind}\,\pi$ of $C^*(H) \times_{\alpha,\,u} K$ is irreducible. This Mackey-type of result (see e.g. [11]) is indeed valid for general twisted crossed products $A \times_{\alpha,\,u} K$. When the 2-cocycle u takes value in the center of A, this was proved by G. Zeller-Meier (see [18]). For completeness, we show in the Appendix (Proposition 5) that Zeller-Meier's result is also true in the general case needed here.

Further, as K is amenable, we also know from [14, Theorem 3.1] that $\mathrm{Ind}\,\pi$ is faithful. Altogether, it follows that $C^*(G)$ has a faithful, irreducible representation, as desired.

□

We can now deduce Theorem 1 from Theorem 2.

Proof of Theorem 1.

Write $G = PSL(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2 = b^3 = 1\rangle$.

Let H denote the kernel of the canonical homomorphism p from G onto $K = \mathbb{Z}_2 \times \mathbb{Z}_3$ ($\simeq \mathbb{Z}_6$).

As mentioned in the introduction, H is a free group on two generators, which may be chosen as

$$x_1 = abab^2 \text{ and } x_2 = ab^2ab.$$

Using Yoshizawa's result mentioned in the Introduction, we may then pick $[\pi] \in \widehat{C^*(H)}^{\,o}$. Set

$$U_1 = i_H(x_1),\, V_1 = \pi(U_1), \quad U_2 = i_H(x_2),\, V_2 = \pi(U_2),$$

so V_1, V_2 are unitary operators acting on the separable Hilbert space $\mathcal{H}_\pi$ on which π acts. As shown by Choi in [6], we may and do assume that V_2 is diagonal relative to some orthonormal basis of $\mathcal{H}_\pi$, with (distinct) diagonal entries given by some $\mu_j \in \mathbb{T}, j \in \mathbb{N}$.

For each $\lambda \in \mathbb{T}$, we let γ_λ be the $*$-automorphism of $C^*(H)$ satisfying

$$\gamma_\lambda(i_H(x_1)) = i_H(x_1),\; \gamma_\lambda(i_H(x_2)) = \lambda\, i_H(x_2)\,.$$

Set $\pi_\lambda = \pi \circ \gamma_\lambda$. Clearly, $[\pi_\lambda] \in \widehat{C^*(H)}^{\,o}$.

We will show that we can pick λ in $\mathbb{T}$ such that $[\pi_\lambda]$ is a free point for the natural action of K on $\widehat{C^*(H)}^{\,o}$. As K is amenable, the primitivity of $C^*(G)$ will then follow from Theorem 2. To see that there exists such a $\lambda \in \mathbb{T}$, we proceed as follows.

As a normalized section for $p : G = \mathbb{Z}_2 * \mathbb{Z}_3 \to K = \mathbb{Z}_2 \times \mathbb{Z}_3$, we choose $n : K \to G$ given by

$$n(i,j) = a^i\, b^j, \quad i \in \{0,1\},\, j \in \{0,1,2\}.$$

For each $k = (i,j) \in K$ we let α_k denote the associated $*$-automorphism of $C^*(H)$, as introduced when defining the natural action of K on $\widehat{C^*(H)}^{\,o}$.

It is clear that $[\pi_\lambda]$ will be a free point for this action of K whenever for each $k \in K, k \neq (0,0)$, we have

$$(\pi_\lambda \circ \alpha_k)(U_r) \not\simeq \pi_\lambda(U_r) \text{ for } r = 1 \text{ or } r = 2.$$

Some elementary computations give:

$$\pi_\lambda(U_1) = V_1,\ \pi_\lambda(U_2) = \lambda V_2;$$

when $k = (0,1)$: $\quad (\pi_\lambda \circ \alpha_k)(U_2) = V_1^*$;

when $k = (0,2)$: $\quad (\pi_\lambda \circ \alpha_k)(U_1) = (\lambda\, V_2)^*$;

when $k = (1,0)$: $\quad (\pi_\lambda \circ \alpha_k)(U_2) = (\lambda\, V_2)^*$;

when $k = (1,1)$: $\quad (\pi_\lambda \circ \alpha_k)(U_2) = V_1$;

when $k = (1,2)$: $\quad (\pi_\lambda \circ \alpha_k)(U_1) = \lambda\, V_2$.

It follows that $[\pi_\lambda]$ will be a free point whenever

$$(*) \quad V_1 \not\simeq \lambda\, V_2,\ V_1 \not\simeq (\lambda\, V_2)^*,\ \lambda\, V_2 \not\simeq (\lambda\, V_2)^*.$$

Now V_2 has non-empty point spectrum $\sigma_p(V_2) = \{\mu_j \mid j \in \mathbb{N}\} \subseteq \mathbb{T}$.

Define $\Omega_1 = \{\, \lambda \in \mathbb{T} \mid V_1 \simeq \lambda\, V_2 \,\}$,
$\Omega_2 = \{\, \lambda \in \mathbb{T} \mid V_1 \simeq (\lambda\, V_2)^* \,\}$,
$\Omega_3 = \{\, \lambda \in \mathbb{T} \mid \lambda V_2 \simeq (\lambda\, V_2)^* \,\}$.

Then Ω_1, Ω_2 and Ω_3 are all countable.

Indeed, if Ω_1 was uncountable, then, as $\sigma_p(V_1) = \lambda\, \sigma_p(V_2)$ for all $\lambda \in \Omega_1$, we would get that $\sigma_p(V_1)$ is uncountable and this is impossible (as $\mathcal{H}_\pi$ is separable). In the same way, we see that Ω_2 must be countable. Finally, if Ω_3 was uncountable, then the equality

$$\lambda\,\{\mu_j \mid j \in \mathbb{N}\} = \bar{\lambda}\,\{\bar{\mu_j} \mid j \in \mathbb{N}\}$$

would hold for uncountably many λ's in $\mathbb{T}$, and this is easily seen to be impossible.

Hence, the set $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ is countable. Especially $\Omega \neq \mathbb{T}$ and $(*)$ holds for every λ in the complement Ω^c of Ω in $\mathbb{T}$. Thus, we have shown that $C^*(PSL(2,\mathbb{Z}))$ is primitive.

Moreover, we shall now show that one can pick a sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ in $\mathbb{T}$ such that $\{\mathrm{Ind}\,\pi_{\lambda_j}\}_{j\in\mathbb{N}}$ is a family of faithful, pairwise unitarily inequivalent, irreducible representations of $C^*(PSL(2,\mathbb{Z}))$.

Consider first $\lambda, \lambda' \in \Omega^c$, so that $\mathrm{Ind}\,\pi_\lambda$ and $\mathrm{Ind}\,\pi_{\lambda'}$ are both irreducible. From the usual criterion for inequivalence of induced irreducible representations (adapted to our setting; see Proposition 5 in the Appendix), $\mathrm{Ind}\,\pi_\lambda$ and $\mathrm{Ind}\,\pi_{\lambda'}$ will be unitarily inequivalent whenever

$$\pi_\lambda \circ \alpha_j \not\simeq \pi_{\lambda'} \quad \text{for all } j \in K\,.$$

Using our previous computations, one sees that this will be satisfied whenever we have

$$V_1 \not\simeq \lambda\, V_2\,,\ V_1 \not\simeq (\lambda\, V_2)^*\,,$$
$$V_1 \not\simeq \lambda'\, V_2\,,\ V_1 \not\simeq (\lambda'\, V_2)^*\,,$$
$$\lambda\, V_2 \not\simeq \lambda'\, V_2\,,\ (\lambda V_2)^* \not\simeq \lambda'\, V_2\,.$$

The first four conditions are always satisfied when $\lambda, \lambda' \in \Omega^c$.

On the other hand, if $\lambda \in \mathbb{T}$ is given and we set

$$\Lambda_\lambda = \{\lambda' \in \mathbb{T} \,|\, \lambda\, V_2 \simeq \lambda'\, V_2 \text{ or } (\lambda V_2)^* \simeq \lambda'\, V_2\}\,,$$

then Λ_λ is countable (arguing as in the first part of the proof). Hence, if we first pick $\lambda \in \Omega^c$ and next pick $\lambda' \in (\Omega \cup \Lambda_\lambda)^c$ (which we may since $\Omega \cup \Lambda_\lambda$ is countable), then all six conditions above are satisfied, and it follows that $\mathrm{Ind}\,\pi_\lambda$ and $\mathrm{Ind}\,\pi_{\lambda'}$ are irreducible, faithful and unitarily inequivalent.

So we start by picking $\lambda_1 \in \Omega^c$. Next, we pick $\lambda_2 \in (\Omega \cup \Lambda_{\lambda_1})^c$. Proceeding inductively, assume that $n \geq 3$ and we have picked $\lambda_j \in (\Omega \cup (\cup_{j=1}^{n-2} \Lambda_{\lambda_j}))^c$ for $j = 2, \ldots, n-1$. Then, as $\Omega \cup (\cup_{j=1}^{n-1} \Lambda_{\lambda_j})$ is countable, we may and do pick $\lambda_n \in (\Omega \cup (\cup_{j=1}^{n-1} \Lambda_{\lambda_j}))^c$.

It is then clear that the family $\{\mathrm{Ind}\, \pi_{\lambda_j}\}_{j \in \mathbb{N}}$ produced in this way has the asserted properties.

$\square$

Remark. Proceeding along the same lines as in the proof of Theorem 1, we have recently been able to use Theorem 2 to prove the following more general result:

$C^*(G)$ is primitive whenever G may be written as the free product $G_1 * G_2$ of two non-trivial amenable groups G_1 and G_2 not both of order 2.

As our proof is quite long and combinatorially involved, we will present the details in a subsequent paper.

$\square$

To prepare for our next result, we recall from [7] that when A is a C*-algebra, one endows the primitive ideal space $\mathrm{Prim}(A)$ with its Jacobson (hull-kernel) topology and $\widehat{A}$ with the weakest topology making the canonical map from $\hat{A}$ onto $\mathrm{Prim}(A)$ continuous.

Groups with Kazhdan's property (T) are thoroughly studied in [5]. It will suffice for us to know that a group G has property (T) when $[\pi_1]$ is isolated in $\widehat{C^*(G)}$, where π_1 denotes the representation of $C^*(G)$ associated with the trivial one-dimensional unitary representation of G.

The following result has apparently not been noticed before.

Proposition 3. *Let G be a group with property (T) and assume that $C^*(G)$ is primitive. Then G is trivial.*

Proof. Set $A = C^*(G)$. As A is primitive, $\{0\} \in \mathrm{Prim}(A)$. Moreover, $\{0\}$ is then dense in $\mathrm{Prim}(A)$.

Pick $[\pi_0] \in \widehat{A}^{\,o}$. Then $\{[\pi_0]\}$ is dense in $\widehat{A}$.

(Indeed, let V be a non-empty open subset of $\widehat{A}$ and let $f : \widehat{A} \to \mathrm{Prim}(A)$ denote the canonical map. Write $V = f^{-1}(W)$ for some non-empty open subset W of Prim (A). Then $\{0\} \in W$, so $\widehat{A}^o = f^{-1}(\{0\}) \subseteq V$. Especially, $[\pi_0] \in V$. It follows that $\overline{\{[\pi_0]\}} = \widehat{A}$).

Now $\{[\pi_1]\}$ is, by assumption, an open subset of $\widehat{A}$. Thus we must have $[\pi_1] = [\pi_0]$. Especially, π_1 must be faithful, which means that G is trivial.

□

Corollary 4. *$PSL(n, \mathbb{Z})$ is not primitive when $n \geq 3$.*

Proof. As $PSL(n, \mathbb{Z})$ has property (T) when $n \geq 3$ (see e.g. [5]), this follows from Proposition 3.

□

Moreover, as $PSL(n, \mathbb{Z})$ is always C*-simple (see [1, 2, 8]), this result shows that C*-simplicity of G does not imply that $C^*(G)$ is primitive.

3 Appendix

We prove here two properties of induced representations of discrete twisted crossed products, which we could not find explicitely in the literature in a form suitable for our purposes.

Let (A, K, α, u) be a twisted C^*-dynamical system as considered by Packer and Raeburn [14], where A is a unital C^*-algebra, K is a discrete group with identity e and (α, u) is a twisted action of K on A; this means that α is a map from K into $\mathrm{Aut}(A)$, the group of $*$-automorphisms of A, and u is a map from $K \times K$ into $\mathcal{U}(A)$, the unitary group of A, satisfying

$$\begin{aligned} \alpha_k \, \alpha_l &= \mathrm{Ad}(u(k,l)) \, \alpha_{kl} \\ u(k,l) \, u(kl,m) &= \alpha_k(u(l,m)) \, u(k,lm) \\ u(k,e) &= u(e,k) = 1 \ , \end{aligned}$$

for all $k, l, m \in K$. (To avoid technicalities, we assume that A is unital; otherwise, one has to assume that the 2-cocycle u takes value in the multiplier algebra of A).

The full twisted crossed product $A \times_{\alpha,u} K$ may then be considered as the enveloping C*-algebra of the Banach $*$-algebra $\ell^1(A, K, \alpha, u)$, which consists of the Banach space $\ell^1(K, A)$ equipped with product and involution given by

$$(f * g)(l) = \sum_{k \in K} f(k)\alpha_k(g(k^{-1}l))u(k, k^{-1}l)$$

$$f^*(l) = u(l, l^{-1})^* \alpha_l(f(l^{-1}))^*$$

$f, g \in \ell^1(K, A)$, $l \in K$.

We let i_K and i_A denote the canonical injections of K and A into $A \times_{\alpha,u} K$, respectively.

Let now π be a non-degenerate representation of A on some Hilbert space $\mathcal{H} = \mathcal{H}_\pi$ and let π_α be the associated representation of A on $\mathcal{H}_K = \ell^2(K, \mathcal{H})$ defined by

$$(\pi_\alpha(a)\xi)(k) = \pi(\alpha_{k^{-1}}(a))\,\xi(k)\ ,\ a \in A, \xi \in \mathcal{H}_K, k \in K\,.$$

For every $k \in K$, let $\lambda_u(k)$ be the unitary operator on $\mathcal{H}_K$ given by

$$(\lambda_u(k)\xi)(l) = \pi(u(l^{-1}, k))\,\xi(k^{-1}l),\ k, l \in K,\ \xi \in \mathcal{H}_K\,.$$

The pair (π_α, λ_u) is then a covariant representation of (A, K, α, u), that is,

$$\pi_\alpha(\alpha_k(a)) = \mathrm{Ad}(\lambda_u(k))(\pi_\alpha(a))$$
$$\lambda_u(k)\lambda_u(l) = \pi_\alpha(u(k, l))\lambda_u(kl)$$

for all $k, l \in K$ and $a \in A$. (Note that we follow [18] here, while the "right" version is used in [14]).

This covariant representation induces a non-degenerate representation $\mathrm{Ind}\,\pi$ of $A \times_{\alpha,u} K$ on $\mathcal{H}_K$ determined by

$$(\mathrm{Ind}\,\pi)(f) = \sum_{k \in K} \pi_\alpha(f(k))\lambda_u(k), \quad f \in \ell^1(K, A)\,,$$

that is, by

$$(\mathrm{Ind}\,\pi)(i_A(a)) = \pi_\alpha(a)\,,\ (\mathrm{Ind}\,\pi)(i_K(k)) = \lambda_u(k)\ ,\ a \in A\,, k \in K\,.$$

For each $k \in K$, let $\mathcal{H}_k$ denote the copy of $\mathcal{H}$ in $\mathcal{H}_K$ given by

$$\mathcal{H}_k = \{\xi \in \mathcal{H}_K \,|\, \xi(l) = 0 \text{ for all } l \in K\,,\ l \neq k\}\,,$$

giving us the natural direct sum decomposition $\mathcal{H}_K = \oplus_{k\in K}\mathcal{H}_k$.

Assume now that π' is a non-degenerate representation of A on $\mathcal{H}'$ and denote by $(\pi'_\alpha, \lambda'_u)$ the associated covariant representation of (A, K, α, u) on $\mathcal{H}'_K$.

Let $T \in \mathcal{B}(\mathcal{H}_K, \mathcal{H}'_K)$. Denote by $[T_{k,l}]_{k,l\in K}$ the matrix of T with respect to the natural direct sum decompositions of $\mathcal{H}_K$ and $\mathcal{H}'_K$, and identify each $T_{k,l}$ as an element in $\mathcal{B}(\mathcal{H}, \mathcal{H}')$.

Hence, if $\eta \in \mathcal{H}$ and $k, l \in K$, then $T_{k,l}\,\eta = (T\,\eta_l)(k)$, where $\eta_l \in \mathcal{H}_K$ is given by $\eta_l(k) = \delta_{k,l}\,\eta$.

Some tedious (but straightforward) computations give:

(1) $\quad (T\,\pi_\alpha(a))_{k,l} = T_{k,l}\,\pi(\alpha_{l^{-1}}(a))\,, \qquad (\pi'_\alpha(a)\,T)_{k,l} = \pi'(\alpha_{k^{-1}}(a))\,T_{k,l}\ ,$

(2) $\quad (T\,\lambda_u(j))_{k,l} = T_{k,jl}\,\pi(u(l^{-1}j^{-1}, j))\,,\ (\lambda'_u(j)\,T)_{k,l} = \pi'(u(k^{-1}, j))\,T_{j^{-1}k,l}\,.$

The following result is due to Zeller-Meier in the case where u takes values in the center of A (see [18, Propositions 3.8 and 4.4]).

Proposition 5. *With assumptions and notation as above, we have:*

a) $\mathrm{Ind}\,\pi$ *is irreducible whenever* π *is irreducible and the stabilizer subgroup* $K_\pi = \{k \in K \mid \pi \circ \alpha_k \simeq \pi\}$ *is trivial.*

b) Assume that π *and* π' *both are irreducible.*

Then $\mathrm{Ind}\,\pi \not\simeq \mathrm{Ind}\,\pi'$ *whenever* $\pi \circ \alpha_j \not\simeq \pi'$ *for all* $j \in K$.

Proof. We begin by proving the following observation:

Assume π *and* π' *are irreducible, and* $\pi \circ \alpha_j \not\simeq \pi'$ *for all* $j \in K$, $j \neq e$.

Let $T \in \mathcal{B}(\mathcal{H}_K, \mathcal{H}'_K)$ *intertwine* $\mathrm{Ind}\,\pi$ *and* $\mathrm{Ind}\,\pi'$.

Then T *is decomposable, that is,* $T_{k,l} = 0$ *for all* $k \neq l$ *in* K, *and* $T_{k,k}$ *intertwines* π *and* π' *for all* $k \in K$.

Indeed, we have $\quad T\,\pi_\alpha(a) = \pi'_\alpha(a)\,T$ for all $a \in A$.

Using this and (1), we get

(3) $\quad T_{k,l}\,\pi(\alpha_{l^{-1}}(a)) = \pi'(\alpha_{k^{-1}}(a))\,T_{k,l}$ for all $k, l \in K$, $a \in A$.

Letting $l = k$, this clearly implies that $T_{k,k}$ intertwines π and π' for all $k \in K$.

Assume now that $k \neq l$. Using (3) with $a = \alpha_k(b)$, we get

(4) $\quad T_{k,l}\,(\pi \circ \mathrm{Ad}(u(l^{-1},k)) \circ \alpha_{l^{-1}k})(b) = (\pi' \circ \mathrm{Ad}(u(k^{-1},k)))(b)\,T_{k,l}$

for all $b \in A$.

From the assumption, we have $\pi' \not\simeq \pi \circ \alpha_{l^{-1}k}$. Hence, it follows that $\pi \circ \mathrm{Ad}(u(l^{-1},k) \circ \alpha_{l^{-1}k}$ and $\pi' \circ \mathrm{Ad}(u(k^{-1},k))$ are irreducible and unitary inequivalent. But (4) says that $T_{k,l}$ intertwines these two representations of A, and we can therefore conclude that $T_{k,l} = 0$.

Hence, we have shown the observation and proceed now with the proof of a) and b).

a) Suppose that π is irreducible and K_π is trivial.

Let $T \in \mathcal{B}(\mathcal{H}_K)$ lie in the commutant of $(\mathrm{Ind}\,\pi)(A \times_{\alpha,u} K)$.

Using the above observation with $\pi' = \pi$, it follows that T is decomposable and $T_{k,k} \in \pi(A)'$ for all $k \in K$. As π is irreducible, this gives that $T_{k,k} \in \mathbb{C}\, I_{\mathcal{H}}$ for all $k \in K$.

Further, we have $T\,\lambda_u(j) = \lambda_u(j)\,T$ for all $j \in K$.

Using this and (2), we get

$$\pi(u(k^{-1},kl^{-1}))\,T_{k,k} = T_{k,k}\,\pi(u(k^{-1},kl^{-1})) = (T\,\lambda_u(kl^{-1}))_{k,l}$$
$$= (\lambda_u(kl^{-1})\,T)_{k,l} = \pi(u(k^{-1},kl^{-1}))\,T_{l,l}\,,$$

which implies that $T_{k,k} = T_{l,l}$ for all $k, l \in K$.

Altogether, this means that T is a scalar multiple of the identity operator on $\mathcal{H}_K$. Hence we have shown that $\mathrm{Ind}\,\pi$ is irreducible, as desired.

b) Assume that π and π' both are irreducible and $\pi \circ \alpha_j \not\simeq \pi'$ for all $j \in K$.

Let $T \in \mathcal{B}(\mathcal{H}_K, \mathcal{H}'_K)$ intertwine $\mathrm{Ind}\,\pi$ and $\mathrm{Ind}\,\pi'$. It follows from the above observation that $T_{k,l} = 0$ for all $k, l \in K$, $k \neq l$, and that $T_{k,k}$ intertwine π and π' for all $k \in K$. As $\pi \not\simeq \pi'$ by assumption, we also have $T_{k,k} = 0$ for all $k \in K$. Hence, $T = 0$. This shows that $\mathrm{Ind}\,\pi \not\simeq \mathrm{Ind}\,\pi'$, as desired.

□

Actually, both implications converse to those stated in a) and b) of Proposition 5 also hold (as in [18]). However, since we don't need these in this paper, we skip the proofs.

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